The Inverses of Toeplitz Band Matrices

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ABSTRACT

The elements of the inverse of a Toeplitz band matrix are given in terms of the solution of a difference equation. The expression for these elements is a quotient of determinants whose orders depend the number of nonzero superdiagonals but not on the order of the matrix. Thus, the formulae are particularly simple for lower triangular and lower Hessenberg Toeplitz matrices. When the number of nonzero superdiagonals is small, sufficient conditions on the solution of the abovementioned difference equation can be given to ensure that the inverse matrix is positive. If the inverse is positive, the row sums can be expressed in terms of the solution of the difference equation.

1. INTRODUCTION

The inversion of Toeplitz band matrices has been considered by many authors. A good summary of results about Toeplitz matrices appeared recently [10]. Formulae for the inverses of general band matrices, including Toeplitz band matrices as a special case, were obtained in [12], and formulae specifically for the inverses of Toeplitz band matrices were derived in [11]. A method using recurrences was described in [3] for the solution of Toeplitz systems of equations. Both [2] and [7] gave methods to calculate the inverses of symmetric Toeplitz band matrices. Two special examples of symmetric Toeplitz band matrices were inverted in [5] and [9]. In this paper Jacobi's theorem is used to develop formulae for the elements of the inverse of a Toeplitz band matrix. The formulae are written in terms of the solution of a difference equation and expressed as a quotient of determinants, where the orders of the determinants depend on the number of nonzero superdiagonals of the original matrix. This compares with [12], in which elements of the inverse of a general band matrix are obtained in terms of determinants whose order depends on the bandwidth, and [11], in which two difference equations are solved to find elements of the inverse. It appears that the formulae given

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here should be useful in cases where there are a small number of superdiagonals but the bandwidth is not necessarily small.

It is often of interest to know that the inverse of a given matrix consists entirely of positive or of nonnegative elements. In [6] a condition depending on the zeros of a polynomial is shown to ensure that the elements of the inverse of a Toeplitz band matrix are nonnegative. Different sufficient conditions that the inverse of a Toeplitz band matrix with a small number of superdiagonals is positive or nonnegative are given here. The conditions are on the solution of the same difference equation which is useful in obtaining the elements of the inverse. Refer to the first example in Section 4 for a Toeplitz band matrix with a nonnegative inverse which does not satisfy the condition in [6] but does satisfy the condition given here. If the inverse is positive or nonnegative, the row sums can generally be obtained in terms of the solution of the difference equation.

The type of matrix to be inverted here has bandwidth r and s superdiagonals. The general form is

$$T_{s,n}=\begin{pmatrix} a_{s+1} & a_s & \cdot & \cdot & \cdot & a_1 & & & & 0 \\ a_{s+2} & a_{s+1} & \cdot & \cdot & \cdot & \cdot & a_1 & & & \\ \cdot & \cdot & \cdot & & & & & \cdot & & \\ \cdot & \cdot & & \cdot & & & & & \cdot & \\ \cdot & \cdot & & & \cdot & & & & & \cdot \\ a_r & \cdot & & & & \cdot & & & & a_1 \\ & a_r & & & & & \cdot & & & \cdot \\ & & \cdot & & & & & \cdot & & \cdot \\ & & & \cdot & & & & & \cdot & \cdot \\ & & & & \cdot & & & & & \cdot \\ 0 & & & & & a_r & \cdot & \cdot & \cdot & a_{s+1} \end{pmatrix}_{n\times n}, \tag{1.1}$$

where $r\leqslant 2n-1$, $0\leqslant s<\min(r,n)$, and a_1 and a_r are both nonzero. Defining

$$A_i=\begin{cases} a_i, & 1\leqslant i\leqslant r,\\ 0 & \text{otherwise,}\end{cases} \tag{1.2}$$

the (i,j) element of $T_{s,n}$ is $A_{i-j+s+1}$.

2. ELEMENTS OF THE INVERSE OF $T_{s,n}$

The matrix $T_{s,n}$ is a submatrix of the lower triangular matrix

$$L=\left(A_{i-j+1}\right)_{n+s\times n+s}.$$

Jacobi's theorem will be used to express $T_{s,n}^{-1}$ in terms of the elements of L^{-1}. The elements of L^{-1} are related to the solution of a difference equation (see [11]). Let g_i be the solution of the homogeneous linear difference equation

$$a_r g_{i-r} + a_{r-1} g_{i-r+1} + \cdots + a_i g_{i-1} = 0, \qquad i \geqslant 3, \tag{2.1}$$

with initial conditions

$$g_1 = \frac{1}{a_1}$$

and

$$g_0 = g_{-1} = \cdots = g_{3-r} = 0 \qquad \text{when} \quad r \geqslant 3,$$

then L^{-1} is the lower triangular matrix $(G_{i-j+1})_{n+s\times n+s}$, where

$$G_i = \begin{cases} g_i, & i \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Jacobi's theorem [1, p. 99] implies that any minor of L equals the complementary signed minor of $(L^{-1})'$ times $\det(L)$. Thus, the (i, j) element of $T_{s,n}^{-1}$, which is the cofactor of the (j, i) element in $T_{s,n}$ divided by the determinant of $T_{s,n}$, is a quotient of minors of L and consequently can be expressed as a quotient of minors of $(L^{-1})'$. The formula for the (i, j) element of $T_{s,n}^{-1}$ is

$$(-1)^s C_{i,j,s,n} / D_{s,n},$$

where

$$C_{i,j,s,n} = \begin{vmatrix} G_i & G_{n+1} & G_{n+2} & \cdots & G_{n+s} \\ G_{i-1} & G_n & G_{n+1} & \cdots & G_{n+s-1} \\ G_{i-2} & G_{n-1} & G_n & \cdots & G_{n+s-2} \\ \vdots & \vdots & \vdots & & \vdots \\ G_{i-s+1} & G_{n-s+2} & G_{n-s+1} & \cdots & G_{n+1} \\ G_{i-j-s+1} & G_{n-j-s+2} & G_{n-j-s+1} & \cdots & G_{n-j+1} \end{vmatrix}_{s+1\times s+1},$$

and

$$D_{s,n}=\begin{vmatrix} G_{n+1} & G_{n+2} & \cdots & G_{n+s} \\ G_n & G_{n+1} & \cdots & G_{n+s-1} \\ \vdots & \vdots & & \vdots \\ G_{n-s+2} & G_{n-s+1} & \cdots & G_{n+1} \end{vmatrix}_{s\times s}.$$

Some examples are illustrated below. The Toeplitz Hessenberg matrix, $T_{1,n}$, has an inverse with elements

$$-\frac{\begin{vmatrix} G_i & G_{n+1} \\ G_{i-j} & G_{n-j+1} \end{vmatrix}}{G_{n+1}}, \tag{2.2}$$

and $T_{2,n}^{-1}$ has elements

$$\frac{\begin{vmatrix} G_i & G_{n+1} & G_{n+2} \\ G_{i-1} & G_n & G_{n+1} \\ G_{i-j-1} & G_{n-j} & G_{n-j+1} \end{vmatrix}}{\begin{vmatrix} G_{n+1} & G_{n+2} \\ G_n & G_{n+1} \end{vmatrix}}. \tag{2.3}$$

It should be noted that when working with determinants in this application the identity

$$|A|\cdot|A(pq,pq)|=\begin{vmatrix} |A(p,p)| & |A(p,q)| \\ |A(q,p)| & |A(q,q)| \end{vmatrix} \tag{2.4}$$

[8, p. 135; 1, p. 48; 3] is often very useful. Here $|A(p,q)|$ is the minor of A when row p and column q are removed, and $|A(pq,pq)|$ is the minor of A when rows p and q and columns p and q are removed.

3. SUFFICIENT CONDITIONS FOR $T_{0,n}^{-1} \geq 0$, $T_{1,n}^{-1} > 0$, AND $T_{2,n}^{-1} > 0$

Sufficient conditions for $T_{0,n}^{-1} \geq 0$, $T_{1,n}^{-1} > 0$, and $T_{2,n}^{-1} > 0$ are given in the theorems which follow.

THEOREM 3.1. *If* $G_i \geqslant 0$ *for* $1 \leqslant i \leqslant n$, *then* $T_{0,n}^{-1} \geqslant 0$.

THEOREM 3.2. *If* $G_i < 0$ *for* $1 \leqslant i \leqslant n+1$ *and*

$$\begin{vmatrix} G_i & G_{i+1} \\ G_{i-1} & G_i \end{vmatrix} > 0 \qquad \textit{for} \quad 2 \leqslant i \leqslant n,$$

then $T_{1,n}^{-1} > 0$.

Proof. The (i, j) element of $T_{1,n}^{-1}$ is given in (2.2), and since $G_{n+1} < 0$, it remains to prove

$$\begin{vmatrix} G_i & G_{n+1} \\ G_{i-j} & G_{n-j+1} \end{vmatrix} > 0 \qquad \text{for} \quad 1 \leqslant i, j \leqslant n.$$

If $i - j \leqslant 0$, then this determinant is $G_i G_{n-j+1}$, which is positive for $1 \leqslant i, j \leqslant n$. The case $i - j \geqslant 1$ is more difficult. The stated conditions imply that

$$\begin{vmatrix} G_{i-j+1} & G_{i-j+2} \\ G_{i-j} & G_{i-j+1} \end{vmatrix} > 0.$$

Now Lemma A.1 in the appendix can be used $n - i$ times and its corollary $j - 1$ times to show that

$$\begin{vmatrix} G_i & G_{n+1} \\ G_{i-j} & G_{n-j+1} \end{vmatrix} > 0 \qquad \text{for} \quad 1 \leqslant i, j \leqslant n. \qquad \blacksquare$$

THEOREM 3.3. *If* $G_i > 0$ *for* $1 \leqslant i \leqslant n+2$,

$$\begin{vmatrix} G_i & G_{i+1} \\ G_{i-1} & G_i \end{vmatrix} > 0 \qquad \textit{for} \quad 2 \leqslant i \leqslant n+1,$$

and

$$\begin{vmatrix} G_i & G_{i+1} & G_{i+2} \\ G_{i-1} & G_i & G_{i+1} \\ G_{i-2} & G_{i-1} & G_i \end{vmatrix} > 0 \qquad \textit{for} \quad 2 \leqslant i \leqslant n,$$

then $T_{2,n}^{-1} > 0$.

Proof. The (i, j) element of $T_{2,n}^{-1}$ is given in (2.3), and since denominator of that expression is positive, it remains to prove that the numerator

$$\begin{vmatrix} G_i & G_{n+1} & G_{n+2} \\ G_{i-1} & G_n & G_{n+1} \\ G_{i-j-1} & G_{n-j} & G_{n-j+1} \end{vmatrix} \tag{3.1}$$

is positive for $1 \leqslant i, j \leqslant n$.

The two main cases to consider are $i - j - 1 \leqslant 0$ and $i - j - 1 > 0$. If $i - j - 1 \leqslant 0$, then this determinant is

$$\begin{vmatrix} G_i & G_{n+1} & G_{n+2} \\ G_{i-1} & G_n & G_{n+1} \\ G_0 & G_{n-j} & G_{n-j+1} \end{vmatrix}. \tag{3.2}$$

When $i = 1$, (3.2) is

$$G_1 \begin{vmatrix} G_n & G_{n+1} \\ G_{n-j} & G_{n-j+1} \end{vmatrix},$$

which can be shown to be positive using methods like those in Theorem 3.2. When $j = n$, (3.2) can also be shown positive. For $i \geqslant 2$ and $j \leqslant n - 1$, start with

$$\begin{vmatrix} G_2 & G_3 & G_4 \\ G_1 & G_2 & G_3 \\ G_0 & G_1 & G_2 \end{vmatrix} > 0$$

and apply Lemma A.2 $n - j - 1$ times and its corollary $i - 2$ times to show (3.2) is positive.

If $i - j - 1 > 0$, then start with

$$\begin{vmatrix} G_{i-j+1} & G_{i-j+2} & G_{i-j+3} \\ G_{i-j} & G_{i-j+1} & G_{i-j+2} \\ G_{i-j-1} & G_{i-j} & G_{i-j+1} \end{vmatrix} > 0$$

and apply Lemma A.2 $n-i$ times and its corollary $j-1$ times to show that (3.1) is positive. It should be noted that the conditions required for Lemma A.2 are met in the two previous cases, as all the relevant 2×2 determinants can be shown positive by methods like those in Theorem 3.2. ■

The row sums of $T_{s,n}^{-1}$ are often required when $T_{s,n}^{-1}\geqslant 0$. They can be expressed as follows. Let

$$S_i=\begin{cases}\sum_{k=1}^{i} G_k, & i\geqslant 1,\\ 0 & \text{otherwise;}\end{cases} \tag{3.3}$$

then the sum of the ith row of $T_{0,n}^{-1}$ is S_i. The sum of the ith row of $T_{1,n}^{-1}$ is

$$\frac{-\begin{vmatrix} G_i & G_{n+1}\\ S_{i-1} & S_n\end{vmatrix}}{G_{n+1}}, \tag{3.4}$$

and the sum of the ith row of $T_{2,n}^{-1}$ is

$$\frac{\begin{vmatrix} G_i & G_{n+1} & G_{n+2}\\ G_{i-1} & G_n & G_{n+1}\\ S_{i-2} & S_{n-1} & S_n\end{vmatrix}}{\begin{vmatrix} G_{n+1} & G_{n+2}\\ G_n & G_{n+1}\end{vmatrix}}. \tag{3.5}$$

For small bandwidth matrices with $a_1+a_2+\cdots+a_r$ not zero, the sum (3.3) is more conveniently expressed another way. Write down the difference equation (2.1) for indices $3,4,5,\ldots,i+1$ and add all the equations. The quantity S_i can thus be expressed

$$S_i=\frac{1+\sum_{k=1}^{r-1}\left(\sum_{l=k+1}^{r} a_l\right)G_{i-k+1}}{\sum_{l=1}^{r} a_l}, \qquad i\geqslant 1. \tag{3.6}$$

4. EXAMPLES

The first example is a lower triangular matrix with $r = 3$, $a_1 = 1$, $a_2 = -1$, $a_3 = -2$, and $s = 0$. For this matrix

$$G_i = \begin{cases} \left[2^i - (-1)^i\right]/3, & i \geqslant 1, \\ 0, & i \leqslant 0. \end{cases}$$

The elements of the inverse are obviously nonnegative. The row sum of the ith row, S_i in (3.6), is

$$S_i = \frac{1 - 3G_i - 2G_{i-1}}{-2} = \frac{2^{i+2} - 3 - (-1)^i}{6}.$$

This example has a nonnegative inverse but does not satisfy the conditions given in [6].

The symmetric Toeplitz tridiagonal has been discussed by many authors, for example [4]. In the notation used here $r = 3$, $a_1 = -1$, $a_2 = \alpha$, $a_3 = -1$, and $s = 1$. Solving the difference equation for $\alpha > 2$,

$$G_i = \begin{cases} -\dfrac{\sinh i\theta}{\sinh\theta} \quad \left(\theta = \cosh^{-1}\dfrac{\alpha}{2}\right), & i \geqslant 1, \\ 0, & i \leqslant 0. \end{cases}$$

The elements of the inverse are given by (2.2). To prove that the inverse elements are all positive, observe $G_i < 0$, $i \geqslant 1$, and

$$\begin{vmatrix} G_i & G_{i+1} \\ G_{i-1} & G_i \end{vmatrix} = 1, \qquad i \geqslant 2,$$

so that Theorem 3.2 applies. The row sums can be obtained from (3.4) by first using (3.6) to find

$$S_i = \frac{1 + (\alpha - 1)G_i - G_{i-1}}{\alpha - 2}.$$

A fairly simple example with $s = 2$ is the symmetric five band matrix in [5]. For that matrix, $r = 5$, $a_1 = 1$, $a_2 = -4$, $a_3 = 6$, $a_4 = -4$, and $a_5 = 1$.

Solving the difference equation,

$$G_i = \begin{cases} \dfrac{i(i+1)(i+2)}{6}, & i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The elements of the inverse are given by (2.3). To see that the inverse is positive, calculate

$$\begin{vmatrix} G_i & G_{i+1} \\ G_{i-1} & G_i \end{vmatrix} = \tfrac{1}{12} i(i+1)^2(i+2),$$

and by (2.4)

$$\begin{vmatrix} G_i & G_{i+1} & G_{i+2} \\ G_{i-1} & G_i & G_{i+1} \\ G_{i-2} & G_{i-1} & G_i \end{vmatrix}$$

$$= \frac{1}{G_i}\left\{ \begin{vmatrix} G_i & G_{i+1} \\ G_{i-1} & G_i \end{vmatrix}^2 - \begin{vmatrix} G_{i-1} & G_i \\ G_{i-2} & G_{i-1} \end{vmatrix} \cdot \begin{vmatrix} G_{i+1} & G_{i+2} \\ G_i & G_{i+1} \end{vmatrix} \right\}$$

$$= \frac{i(i+1)(i+2)}{6}.$$

Now Theorem 3.3 can be applied to show that the inverse is positive.

APPENDIX

Two lemmas will be established in this appendix.

LEMMA A.1. *If*

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} > 0$$

with $a_1 b_2 \geq 0$, $a_2 b_3 \geq 0$, *and* $a_2 b_2 > 0$, *then*

$$\begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix} > 0.$$

Proof. The first two inequalities can be written

$$a_2b_1 > a_1b_2 \quad \text{and} \quad a_3b_2 > a_2b_3.$$

Multiplication of the respective left and right sides of these inequalities gives

$$a_2a_3b_1b_2 > a_1a_2b_2b_3$$

which upon division by a_2b_2 yields the required result. ■

COROLLARY. *Interchanging the diagonal elements gives a similar result that*

$$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} a_3 & b_3 \\ a_2 & b_2 \end{vmatrix} > 0$$

with $a_1b_2 \geqslant 0$, $a_2b_3 \geqslant 0$, *and* $a_2b_2 > 0$ *implies*

$$\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} > 0.$$

LEMMA A.2. *If*

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} c_2 & c_3 & c_4 \\ b_2 & b_3 & b_4 \\ a_2 & a_3 & a_4 \end{vmatrix} > 0$$

with

$$c_3 > 0, \qquad \begin{vmatrix} c_1 & c_3 \\ a_1 & a_3 \end{vmatrix} \cdot \begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix} \geqslant 0,$$

$$\begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix} \cdot \begin{vmatrix} c_3 & c_4 \\ b_3 & b_4 \end{vmatrix} \geqslant 0, \quad \text{and} \quad \begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix} \cdot \begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix} > 0,$$

then

$$\begin{vmatrix} c_1 & c_3 & c_4 \\ b_1 & b_3 & b_4 \\ a_1 & a_3 & a_4 \end{vmatrix} > 0.$$

Proof. The determinant relation (2.4) can be used to rewrite these 3×3 determinants. The proof which then follows is very similar to the proof used for Lemma A.1.

Rotating the columns so that c_3 is in the top left corner and applying (2.4) with $p=2$, $q=3$, it can be shown that

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \frac{1}{c_3}\left(\begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix}\cdot\begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix} - \begin{vmatrix} c_1 & c_3 \\ a_1 & a_3 \end{vmatrix}\cdot\begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix}\right). \tag{A.1}$$

Rotating the columns so that c_3 is in the top left corner and applying (2.4) with $p=2$, $q=3$, it can be shown that

$$\begin{vmatrix} c_2 & c_3 & c_4 \\ b_2 & b_3 & b_4 \\ a_2 & a_3 & a_4 \end{vmatrix} = \frac{1}{c_3}\left(-\begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix}\cdot\begin{vmatrix} c_3 & c_4 \\ b_3 & b_4 \end{vmatrix} + \begin{vmatrix} c_3 & c_4 \\ a_3 & a_4 \end{vmatrix}\cdot\begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix}\right). \tag{A.2}$$

Finally, rotating columns so that c_3 is in the top left corner and applying (2.4) with $p=2$, $q=3$,

$$\begin{vmatrix} c_1 & c_3 & c_4 \\ b_1 & b_3 & b_4 \\ a_1 & a_3 & a_4 \end{vmatrix} = \frac{1}{c_3}\left(-\begin{vmatrix} c_1 & c_3 \\ a_1 & a_3 \end{vmatrix}\cdot\begin{vmatrix} c_3 & c_4 \\ b_3 & b_4 \end{vmatrix} + \begin{vmatrix} c_3 & c_4 \\ a_3 & a_4 \end{vmatrix}\cdot\begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix}\right). \tag{A.3}$$

The right sides of (A.1) and (A.2) can be written as inequalities in a manner similar to that used in the proof of Lemma A.1. Multiplication of the respective left and right sides of these inequalities and division by

$$\frac{1}{c_3}\begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix}\cdot\begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix}$$

shows that the determinant (A.3) is positive. ∎

COROLLARY. *If*

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} > 0 \quad \textit{and} \quad \begin{vmatrix} d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} > 0$$

with

$$c_1 > 0, \qquad \begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix} \geqslant 0,$$

$$\begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} d_1 & d_3 \\ c_1 & c_3 \end{vmatrix} \geqslant 0, \quad \textit{and} \quad \begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix} > 0,$$

then

$$\begin{vmatrix} d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} > 0.$$

I am indebted to the referee for suggesting the use of Jacobi's theorem to derive the formula for $T_{s,n}^{-1}$ in Section 2. This is a great improvement over my original proof. I also thank A. L. Andrew for bringing Ref. [3] to my attention.

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